Speed Round Solutions

LMT Fall 2024

December 14, 2024

1. **[6]** Find the value of
$$(2+0+2+4)+\left(2^0+2^4\right)+\left(2^{0^{2^4}}\right).$$

   Proposed by: Atticus Oliver

   Solution. $\boxed{26}$

   Evaluate to get $8+1+16+1=\boxed{26}$. □

2. **[6]** The angles in triangle ABC are such that $\angle A$, $\angle B$, $\angle C$ form an arithmetic progression in that order. Find the measure of $\angle B$, in degrees.

   Proposed by: Muztaba Syed

   Solution. $\boxed{60}$

   The sum of the three angles is 180, and the average is 60. Also $\angle B$ is the average, so the answer is $\boxed{60}$. □

3. **[6]** High schoolers chew a lot of gum. At the supermarket, 15 packs of 14 sticks of gum costs \$10. If 1400 high schoolers chew 3 sticks of gum per day, find the total number of dollars spent by these high schoolers on gum per week.

   Proposed by: Lena Lee

   Solution. $\boxed{1400}$

   The number of sticks chewed is $1400\cdot 3\cdot 7$. We can multiply this by the cost per stick of gum, which is $\frac{10}{15\cdot 14}$, so the answer is
$$\frac{1400\cdot 3\cdot 7\cdot 10}{15\cdot 14}=\boxed{1400}.$$
□

4. **[6]** Define $x\star y$ to be $xy\cdot\min(x,y)$ and $x\diamond y$ to be $xy\cdot\max(x,y)$. Suppose $ab=4$. Find the value of
$$(a\star b)\cdot(a\diamond b).$$

   Proposed by: Muztaba Syed

   Solution. $\boxed{64}$

   Note that $\min(a,b)\cdot\max(a,b)=ab$. So we just need to find the value of $a^3b^3=(ab)^3=4^3=\boxed{64}$. □

5. **[6]** Find the area of the quadrilateral with vertices at $(0,0),(2,0),(20,24),(0,2)$ in that order.

   Proposed by: Muztaba Syed

   Solution. $\boxed{44}$

   Connecting the points $(0,0)$ and $(20,24)$, this is just 2 triangles with base 2 and heights 20 and 24. This gives us an answer of $\boxed{44}$. □

6. **[6]** Danyang is doing math. He starts to draw an isosceles triangle, but only manages to draws an angle of $70°$ before he has to leave for recess. Find the sum of all possible values for the smallest angle in Danyang's triangle.

   Proposed by: Jonathan Liu

   Solution. $\boxed{95}$

   The triangle is either a $70-70-40$ triangle or a $70-55-55$ triangle. The answer is then $40+55=\boxed{95}$. □

7. **[6]** Find the sum of the distinct prime factors of 512512.

   Proposed by: Rohan Danda

   Solution. $\boxed{33}$

   We can write this as

$$512\cdot 1001 = 2^9\cdot 7\cdot 11\cdot 13 \implies 2+7+11+13=\boxed{33}.$$

□

8. **[6]** The LHS Math Team is doing Karaoke. William sings every song, David sings every other song, Peter sings every third song, and Muztaba sings every fourth song. If they sing 600 songs, find the average number of people singing each song.

   Proposed by: Edwin Zhao

   Solution. $\boxed{\frac{25}{12}}$

   William sings 600 songs, David sings 300 songs, Peter sings 200 songs, and Muztaba sings 150 songs. Thus, we have $\frac{600+300+200+150}{600}=\boxed{\frac{25}{12}}$. □

9. **[6]** Find the median of the positive divisors of 6^4-1.

   Proposed by: Muztaba Syed

   Solution. $\boxed{36}$

   We can write $6^4-1=(6^2-1)(6^2+1)=35\cdot 37$. $(35,37)$ is the middle factor pair of this number, and their average is $\boxed{36}$. □

10. **[6]** Today is 12/14/24, which is of the form $ab/ac/bc$ for not necessarily distinct digits a, b, and c. Find the number of other dates in the 21st century that can also be written in this form.

    Proposed by: Adam Ge

    Solution. $\boxed{110}$

    From the month part of the date, we get that the possible (a,b) are $(0,1)$, $(0,2)$, $\dots$, $(1,0)$, $(1,1)$, $(1,2)$.

    c can be any digit, and a and c can never be 0 simultaneously. Therefore, there are $9\cdot 9+3\cdot 10=111$ possible dates, $\boxed{110}$ not including 12/14/24. □

11. **[6]** Let x and y be real numbers such that

$$x+\frac{1}{y}=20 \text{ and } y+\frac{1}{x}=24.$$

Find $\frac{x}{y}$.

Proposed by: Henry Eriksson

Solution. $\boxed{\frac{5}{6}}$

Multiply the first equation by y and the second by x to find that

$$xy + 1 = 20y,$$
$$xy + 1 = 24x.$$

So $20y = 24x \implies \frac{x}{y} = \boxed{\frac{5}{6}}$. □

12. **[6]** Call a number *orz* if it is a positive integer less than 2024. Call a number *admitting* if it can be expressed as $a^2 - 1$ where a is a positive integer. Finally call a number *muztaba* if it has exactly 4 positive integer factors. Find the number of *muztaba admitting orz* numbers.

Proposed by: Edwin Zhao

Solution. $\boxed{7}$

If a number is *muztaba*, it's either expressible as $p_1 \cdot p_2$ where p_1 and p_2 are primes or it can be expressed as p^3 where p is prime. $a^2 - 1$ can also be expressed as $(a+1)(a-1)$. Therefore, either $a+1$ and $a-1$ are both prime numbers (resulting in the first case) or $a+1$ is a prime number squared and $a-1$ is the prime number in question.

In the first case, we need to find primes that are two apart. There are 6 of these pairs that result in their product being less than 2023 (41 and 43, 29 and 31, 17 and 19, 11 and 13, 5 and 7, 3 and 5). However, we also need to add one possibility from the second case ($3^2 - 1 = 8$, which has 4 factors). Therefore, our final answer is $6 + 1 = \boxed{7}$. □

13. **[6]** Some math team members decide to study at Cary Library after school. They walk 6 blocks north, then 8 blocks west to get there. If they walk n blocks east from the library, they can buy boba from CoCo's. If CoCo's is the same distance from school as it is from the library, find n.

Proposed by: Lena Lee

Solution. $\boxed{\frac{25}{4}}$

We will use similar triangles. The school is LHS. CoCo's lies on the perpendicular bisector of LHS and Cary, so the triangle formed by LHS, Cary, and 6 blocks north of LHS is similar to the triangle formed by CoCo, Cary, and the midpoint of Cary and LHS. So the answer is $\frac{5}{4} \cdot 5 = \boxed{\frac{25}{4}}$. □

14. **[6]** Isabella assigns a distinct integer from 1 to 6 to each row and column of a 3×3 grid. In each entry, she writes either the sum or the product of the values assigned to the corresponding row and column. Find the maximum possible value of the sum of all entries in the grid.

Proposed by: Muztaba Syed

Solution. $\boxed{113}$

Note that we should always do the product unless one of the values is 1 since

$$ab \geq a + b \iff (a-1)(b-1) \geq 1,$$

which is true when both a and b are greater than 1. If the values assigned to the rows are a, b, c, and the columns are d, e, f, then the total sum is (since if $a = 1$ then $a + b = ab + 1$)

$$(a+b+c)(d+e+f) + 3.$$

Since $a+b+c+d+e+f = 21$, this is maximized at $11 \cdot 10 + 3 = \boxed{113}$. □

15. **[6]** Find the value of $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + \cdots + 6 \cdot 7 \cdot 8 \cdot 9$.

Proposed by: Jonathan Liu

Solution. $\boxed{6048}$

This is equivalent to $24 \cdot \left[\binom{9}{4} + \binom{8}{4} + \ldots\ldots + \binom{4}{4}\right] = 24 \cdot \binom{10}{5} = \boxed{6048}$ (using the Hockey-Stick Identity). □

16. **[6]** Let $ZHAO$ be a square with area 2024. Let X be the center of this square and let C, D, E, K be the centroids of XZH, XHA, XAO, and XOZ, respectively. Find $[ZHAO] + [CZHAO] + [DZHAO] + [EZHAO] + [KZHAO]$.

(Here $[\mathscr{P}]$ denotes the area of the polygon $\mathscr{P}$.)

Proposed by: Christopher Cheng

Solution. $\boxed{8096}$

$CZHAO$ and $EZHAO$ obviously each are missing $1/4$ of $ZHAO$, and by symmetry $KZHAO$ and $DZHAO$ are missing $1/2$ of $ZHAO$ combined so the sum is just $4 \cdot [ZHAO] + [ZHAO] - [ZHAO] = 4 \cdot [ZHAO] = 4 \cdot 2024 = \boxed{8096}$. □

17. **[6]** For positive integers x, let

$$f(x) = \begin{cases} \frac{f\left(\frac{x}{2}\right)}{2} & \text{if } x \text{ is even,} \\ 2^{-x} & \text{if } x \text{ is odd.} \end{cases}$$

Find $f(1) + f(2) + f(3) + \ldots$.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{4}{3}}$

First we can evaluate the sum of $f(n)$ for all odd terms. This is a geometric sequence with first term $\frac{1}{2}$ and common ratio $\frac{1}{4}$, thus the sum is $\frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$. Notice that the sum of $f(n)$ for all terms with one factor of two is half this sum, the sum of $f(n)$ for all terms with two factors of two is one fourth this sum, and so on. Thus we calculate the answer to be

$$\frac{2}{3} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) = \frac{2}{3} \cdot \frac{1}{1 - \frac{1}{2}} = \boxed{\frac{4}{3}}.$$

□

18. **[6]** Find the number of ways to split the numbers from 1 to 12 into 4 non-intersecting sets of size 3 such that each set has sum divisible by 3.

Proposed by: Muztaba Syed

Solution. $\boxed{640}$

A sum is divisible by 3 if the numbers are all the same mod 3 or all distinct. If there exists a group which has all three the same, then this forces there to be a group with 0s, a group with 1s, a group with 2s, and a group with $(0,1,2)$ (values taken mod 3). There are $4^3 = 64$ ways to assign this.

Otherwise every group has all values distinct mod 3. There are $(4!)^2$ ways to assign this. In total our answer is $64 + 576 = \boxed{640}$. □

19. **[6]** Let $P(n)$ denote the product of digits of n. Find the number of positive integers $n \le 2024$ where $P(n)$ is divisible by n.

Proposed by: James Wu

Solution. $\boxed{485}$

When n only has one digit, $P(n) = n$ is divisible by n.

For any n that has more than one digits, let d be the number of digits and represent n with $10^{d-1} \cdot a + b$ where $b < 10^{d-1}$. We know that $P(b) \le 9^{d-1} < 10^{d-1}$. Therefore, we have $a \cdot P(b) < a \cdot 10^{d-1} \le 10^{d-1} \cdot a + b$.

We are now left with two cases that could possibly satisfy the condition: n has one digit and $P(n) = 0$. There are 9 positive integers n that have one digit. All integers from 2000 to 2024 contain zeroes, so the number of positive integers $n \le 2024$ with $P(n) \ne 0$ is $9 + 9^2 + 9^3 + 1 \cdot 9^3 = 1548$. The number of positive integers $n \le 2024$ with $P(n) = 0$ is $2024 - 1548 = 476$.

The two cases don't overlap, so the answer is $9 + 476 = \boxed{485}$. □

20. **[6]** Henry places some rooks and some kings in distinct cells of a 2×8 grid such that no two rooks attack each other and no two kings attack each other. Find the maximum possible number of pieces on the board.

(Two rooks *attack* each other if they are in the same row or column and no pieces are between them. Two kings attack each other if their cells share a vertex.)

Proposed by: Muztaba Syed

Solution. $\boxed{9}$

Split the grid into 4 2×2 grids. Each of these can contain at most 1 king, so there are at most 4 kings.

If we try 3 kings it is easy to see we can place 8 pieces since in each row the number of rooks is $\leq 1+$(number of kings). When we place 4 kings one of them must be on the leftmost or rightmost column, so 10 won't work. We can easily construct $\boxed{9}$ by alternating rooks and kings in one row and placing a single rook in the other.

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♖	♔	♖	♔	♖	♔	♖	♔

□

21. **[6]** Let ABC be a triangle with $\angle ABC = 90^\circ$. Let D and E be the feet from B and C to the median from A, respectively. Suppose $DE = 4$ and $CD = 5$. Find the area of ABC.

Proposed by: Muztaba Syed

Solution. $\boxed{\dfrac{39}{2}}$

Let M be the midpoint of BC. Note that $CE = 3$ by the Pythagorean Theorem. Then also note that $BD = CE$ since triangles ABM and AMC have the same area. So we see $\triangle BDM \cong \triangle CEM$, so $ME = 2$ and $MC = MB = \sqrt{13}$. Then since $ABEC$ is cyclic by Power of a Point at M we see $AM = \frac{13}{2}$. So the answer is (using AM as a base) $\frac{13}{2} \cdot 3 = \boxed{\dfrac{39}{2}}$. □

22. **[6]** Chris has a list of 5 distinct numbers and every minute he independently and uniformly at random swaps a pair of them. Find the probability that after 4 minutes the order of the list is the same as the original list.

Proposed by: Samuel Tsui

Solution. $\boxed{\dfrac{17}{500}}$

Case 1: Back to original arrangement after two swaps. This gives $\frac{1}{100}$.

Case 2: Two disjoint pairs of swaps. There are 10 options for the first swap, and 3 for the next. These can happen in 2 orders for the next 2 swaps. This gives $\frac{6}{1000}$.

Case 3: Swapping in a cycle. There are 10 ways to choose our first swap, followed by $2 \cdot 3$ options for the next swap. From here there are 3 ways to finish. This gives $\frac{18}{1000}$.

Thus the total probability is $\boxed{\dfrac{17}{500}}$. □

23. **[6]** Circles ω_1 and ω_2 intersect at points X and Y. The common external tangent of the two circles closer to X intersects ω_1 and ω_2 at A and B, respectively. Given that $AB = 6$, the radius of ω_1 is 3, and AY is tangent to ω_2, find XY^2.

Proposed by: Jerry Xu

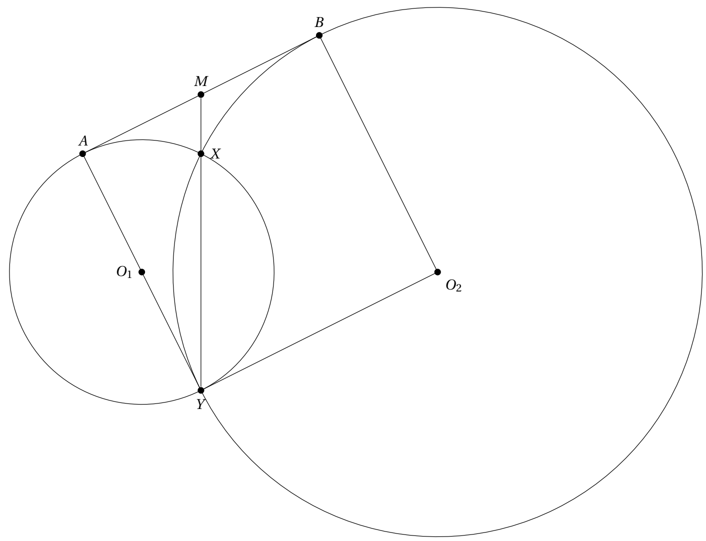
Solution. $\boxed{\dfrac{144}{5}}$

Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively, and let r_2 be the radii of ω_2. AY is twice the radius of ω_1, so it is a diameter. Then, we have $AB \perp BO_2$, $O_2Y \perp YA$, $YA \perp AB$, and $r_2 = BO_2 = YO_2$, so ABO_2Y is a square. Then $r_2 = AB = 6$ and $r_1 = \frac{YA}{2} = 3$.

Now, let $M = \overline{XY} \cap \overline{AB}$. Since M lies on the radical axis of ω_1 and ω_2, $MA^2 = MB^2$, i.e. $MA = MB = 3$. Then, we have by Pythagorean Theorem that $XY = \sqrt{YA^2 + MA^2} = 3\sqrt{5}$. Let $MX = x$. By PoP at M wrt either circle, we have

$$3x\sqrt{5} = 9 \implies x = \frac{3\sqrt{5}}{5}.$$

Hence, $XY = 3\sqrt{5} - x = \frac{12\sqrt{5}}{5}$. Our answer is $\boxed{\dfrac{144}{5}}$.



□

24. **[6]** Find the number of positive integers x that satisfy

$$\left\lfloor \frac{2024}{\left\lfloor \frac{2024}{x} \right\rfloor} \right\rfloor = x.$$

Proposed by: James Wu

Solution. $\boxed{88}$

Let $y = \left\lfloor \frac{2024}{x} \right\rfloor$. We note that from the given equation we also have

$$\left\lfloor \frac{2024}{\left\lfloor \frac{2024}{x} \right\rfloor} \right\rfloor = \left\lfloor \frac{2024}{y} \right\rfloor = x.$$

As $\lfloor \sqrt{2024} \rfloor = 44$, we can consider the cases $x \le 44$ and $x > 44$ separately. Notice that when $x > 44$, $y \le 44$, so the number of $x > 44$ that satisfy the condition will be the same as $x \le 44$. This means that the answer is just double the count for $x \le 44$. By the definition of the floor function, we can write the inequality

$$y \le \frac{2024}{x} \le y + 1.$$

Multiply by x to get

$$xy \leq 2024 \leq xy + x < xy + y = y(x+1).$$

Then, we divide by y to obtain

$$x \leq \frac{2024}{y} < x+1,$$

which means that $\left\lfloor \frac{2024}{y} \right\rfloor = x$. Therefore, all positive integers that satisfy $x \leq 44$ satisfy the condition. We multiply the number by two, so our answer is $44 \cdot 2 = \boxed{88}$. □

25. **[6]** Let a_n be a sequence such that $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = \frac{a_{n+1}a_n}{a_{n+1}+a_n}$. Find the value of

$$\sum_{n=1}^{\infty} \frac{1}{a_n 3^n}.$$

Proposed by: Samuel Tsui and Jonathan Liu

Solution. $\boxed{\frac{3}{5}}$

Let the sum be S. We get

$$S = \frac{1}{3} + \frac{1}{3^2} + \sum_{n=3}^{\infty} \frac{1}{a_n 3^n} = \frac{4}{9} + \sum_{n=3}^{\infty} \frac{1}{a_{n-1}3^n} + \frac{1}{a_{n-2}3^n} = \frac{4}{9} + \frac{1}{3}\sum_{n=3}^{\infty} \frac{1}{a_{n-1}3^{n-1}} + \frac{1}{9}\sum_{n=3}^{\infty} \frac{1}{a_{n-2}3^{n-2}}.$$

Thus $S = \frac{4}{9} + \frac{1}{3}\left(S - \frac{1}{3}\right) + \frac{1}{9}S$ and solving gives $S = \boxed{\frac{3}{5}}$.

In fact rewriting the condition as $\frac{1}{a_{n+2}} = \frac{1}{a_{n+1}} + \frac{1}{a_n}$ and defining $b_n = \frac{1}{a_n}$, we see that b_n is just the Fibonacci Sequence. So $a_n = \frac{1}{F_n}$. □